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On a Geometric Locus in Taxicab Geometry

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Abstract. In axiomatic geometry, the taxicab model of geometry is important as an example of a geometry where the SAS Postulate does not hold. Some properties that hold true in Euclidean geometry are not true in taxicab geometry. For this reason, it is important to understand what happens with various classes of geometric loci in taxicab geometry. In the present study, we focus on a geometric locus question inspired by a problem originally posed by Ţiţeica in the Euclidean context; our study presents the solution to this question in the taxicab plane.

1. Introduction

The taxicab geometry is particularly important in foundations of geometry because it provides an example of geometry where the Side Angle Side Postulate does not hold (see e.g. [8]). In the recent decades, several investigations have focused on various properties of taxicab geometry, some of them inspired from questions studied in advanced Euclidean geometry (see e.g. [2, 4, 6]). An introduction in the fundamental properties of taxicab geometry is [3].

A well-known reference in advanced Euclidean geometry is Ţiţeica's problem book [7]. For the historical context in which the problem book [7] was written and on Ţiţeica's research, including his doctoral dissertation written under Gaston Darboux's direction, see [1]. The problem book [7] is cited by many authors and motivated many contemporary problems. In the present work, we will focus on one particular question, namely, Problem 143. We will ask the question not in the Euclidean context, but in the context of taxicab geometry.

The taxicab distance between two points (x_1, y_1) and (x_2, y_2) in the Cartesian plane is defined (see e.g. [8], p. 39) by

$$\rho((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|.$$

A direct verification shows that ρ is a metric. The Cartesian plane endowed with the metric induced by the distance ρ yields the taxicab geometry.

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2. Right triangle leg ratio

Many relationships in Euclidean geometry do not hold in taxicab geometry. A well-known example is that SAS congruence fails in the taxicab plane; another is that the area of a triangle cannot simply be expressed in the classic $\frac{1}{2}bh$ (see [2]). Nonetheless, a handful of relationships do remain valid in the taxicab plane. For example, we present the following proposition.

Proposition 1. *The ratio between the two legs of a right triangle in the taxicab plane is equal to the ratio between the same two legs in the Euclidean plane.*

Proof. Let a and b denote the legs of the right triangle. We denote the taxicab lengths of a and b by a_T and b_T, and we denote the Euclidean lengths of a and b by a_E and b_E. If a and b are parallel to the coordinate axes, then $a_T = a_E$ and $b_T = b_E$, so clearly $\frac{a_T}{b_T} = \frac{a_E}{b_E}$. Otherwise, let a have nonzero slope m; this means that b has slope $-\frac{1}{m}$. We have the relations

$$a_T = \frac{1+|m|}{\sqrt{1+m^2}} \cdot a_E,$$

$$b_T = \frac{1+|-\frac{1}{m}|}{\sqrt{1+(-\frac{1}{m})^2}} \cdot b_E = \frac{1+|m|}{\sqrt{1+m^2}} \cdot b_E$$

(see [2]). Dividing these two expressions yields $\frac{a_T}{b_T} = \frac{a_E}{b_E}$. □

3. A novel locus

The problem book [7] is cited by many authors and has motivated many contemporary problems. For example, V. Pambuccian's work [5] incorporates the axiomatic analysis of a problem found originally in Ţiţeica's problem book. Pursuing a similar idea, we now examine what happens to Problem 143, which was originally stated in [7] in the Euclidean context, if we study it in taxicab geometry. Thus, we ask the following:

Question. Consider a circle with center O and radius r in the taxicab plane. Point A is located within the circle. Find the locus of midpoints of all chords of the circle that pass through A.

In Euclidean geometry, the locus is well-known. It is simply a circle with diameter OA. On the other hand, when we consider this same locus problem in the context of taxicab geometry, we quickly see that the locus is not so simple. Figure 1 shows an example of such a locus:

Theorem 2. *In general, the locus of midpoints of chords that pass through a point A consists of two straight line segments and two hyperbolic sections, at least one of which contains A.*

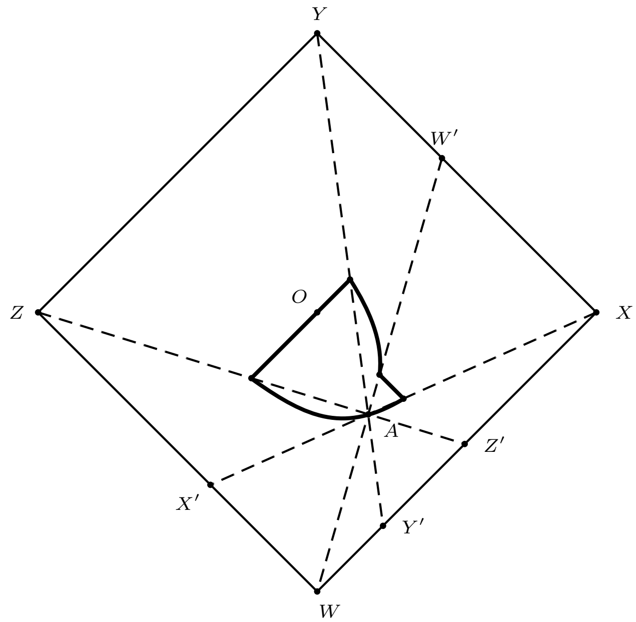

Figure 1

3.1. *Set-Up for proof.* Without loss of generality, we place O at the origin, and let $A = (x_A, y_A)$ such that $x_A \geq 0$ and $y_A \leq -x_A$. We can make that last assumption due to the symmetry of the taxicab plane; reflecting the circle across the axes and the lines $y = \pm x$ essentially preserves the shape of the locus. In other words, given any point A within the circle, we can reflect that point and the specified locus about the axes and the lines $y = \pm x$ until the image of A satisfies $x_A \geq 0$ and $y_A \leq -x_A$.

With the notations specified above, let the vertices of the circle be labeled X, Y, Z, and W, where X lies on the x-axis and the vertices are labeled counterclockwise. Particularly, $X = (r, 0)$, $Y = (0, r)$, $Z = (-r, 0)$, and $W = (0, -r)$. Furthermore, let XA, YA, ZA, and WA intersect the circle again at X', Y', Z', and W', respectively. Then we claim that the locus consists of four parts:

(1) The locus of midpoints of the chords between XX' and $W'W$ is a straight line from the midpoint of XX' to the midpoint of $W'W$ along the line $y = -x$.

(2) The locus of midpoints of the chords between YY' and ZZ' is a straight line from the midpoint of YY' to the midpoint of ZZ' along the line $y = x$.

(3) The locus of midpoints of the chords between $W'W$ and YY' is a hyperbolic section from the midpoint of $W'W$ to the midpoint of YY' that is centered at $(\frac{x_A+r}{2}, \frac{y_A}{2})$.

(4) The locus of midpoints of the chords between ZZ' and $X'X$ is a hyperbolic section from the midpoint of ZZ' to the midpoint of $X'X$ that is centered at $(\frac{x_A}{2}, \frac{y_A-r}{2})$. Point A lies on this hyperbolic section.

3.2. *Proof of Theorem 2.*

(1) and (2): Consider any chord passing through A between XX' and $W'W$. Since the endpoints of this chord lie on parallel lines, the midpoint of the chord must be halfway between these lines, also forming a parallel line with the two original ones. Particularly, the line that the midpoints fall on is $y = -x$, from the midpoint of XX' to the midpoint of $W'W$. By the same reasoning, the locus of midpoints of any chord passing through A between YY' and $Z'Z$ is on $y = x$, from the midpoint of YY' to the midpoint of $Z'Z$.

(3) and (4): We prove (4); then (3) follows by a similar argument.

The equations of lines ZW and WX are $y = -x-r$ and $y = x-r$, respectively. Consider the endpoint C on ZW of a chord passing through A with coordinates $(x_C, -x_C - r)$. The point D at which CA intersects WX is uniquely determined, so we can calculate D using the slope of CA and hence the equations of lines CA and WX:

$$D = \left(\frac{x_A x_C + r x_C + y_A x_C}{y_A - x_A + 2x_C + r}, \frac{x_A x_C + y_A x_C - r y_A + r x_A - r x_C - r^2}{y_A - x_A + 2x_C + r}\right).$$

The midpoint M of chord CD is simply the average of the points C and D:

$$M = \left(x_C + \frac{x_C(x_A - x_C)}{y_A - x_A + 2x_C + r}, \frac{x_C(x_A - x_C)}{y_A - x_A + 2x_C + r} - r\right).$$

To find the locus of midpoints with the above expression, we wish to relate the x and y coordinates. We set

$$x = x_C + \frac{x_C(x_A - x_C)}{y_A - x_A + 2x_C + r} = \frac{x_C y_A + x_C^2 + r x_C}{y_A - x_A + 2x_C + r},$$
$$y = \frac{x_C(x_A - x_C)}{y_A - x_A + 2x_C + r} - r = \frac{x_C x_A - x_C^2 - 2r x_C - r y_A + r x_A - r^2}{y_A - x_A + 2x_C + r}.$$

These equations yield

$$x + y = \frac{x_C y_A + x_C x_A - r x_C - r y_A + r x_A - r^2}{y_A - x_A + 2x_C + r},$$
$$x - y = x_C + r.$$

Multiplying through by the denominator and substituting $x_C = x - y - r$ yields

$$\begin{aligned} &(x+y)(y_A - x_A + 2(x-y-r) + r) \\ &= (x-y-r)y_A + (x-y-r)x_A - r(x-y-r) - r y_A + r x_A - r^2 \\ \Longrightarrow\ & x^2 - y^2 - x x_A + y y_A - yr + y_A r = 0 \\ \Longrightarrow\ & \left(x - \frac{x_A}{2}\right)^2 - \left(y - \frac{y_A - r}{2}\right)^2 = \frac{x_A^2}{4} - \frac{(y_A + r)^2}{4}. \end{aligned}$$

This is precisely the form of a hyperbola, with center $(\frac{x_A}{2}, \frac{y_A - r}{2})$, as desired. Clearly, the point $A = (x_A, y_A)$ lies on this hyperbolic section since

$$\left(x_A - \frac{x_A}{2}\right)^2 - \left(y_A - \frac{y_A - r}{2}\right)^2 = \frac{x_A^2}{4} - \frac{(y_A + r)^2}{4}.$$

A similar argument proves (3) as well. This completes the proof of Theorem 2.

Remarks. (1) In some cases, such as when A lies on the axes or $y = \pm x$ lines, some of these segments or sections may be degenerate. For example, if A lies on either of the coordinate axes, the locus consists of two straight line segments and one hyperbolic section. If A lies on $y = x$ or $y = -x$, the locus consists of one straight line segment and two hyperbolic sections. Clearly, if A is at the origin, the locus is simply a point.

(2) In advanced Euclidean geometry, we work within the axiomatic context given by the postulates of Euclidean geometry, which itself can be viewed in many axiomatic contexts (see [8]). Our present study points out how much a geometric locus can change in an axiomatic framework where the SAS Postulate does not hold any longer.

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